Problem Chosen	2025	Team Control Number
A	MCM/ICM	2527769
	Summary Sheet	

Summary

Given a set of stairs of archaeological interest, our model aims to look at the pattern of erosion to determine the total number of footsteps those stairs have endured over the centuries. We present a linear model that accounts for the amount of weathering the steps have endured from environmental factors. The model uses a linear relation between the number of steps taken on the stairs and the amount of wear measured by approximating the eroded area as a quarter-ellipsoid. We assume the stairs are made out of granite, with a Mohs hardness rating of 6.2, and we assume the archaeologists have a rough estimate of the age of the stairs. From this information, our model can calculate how much of the existing erosion was caused by nature and how much was caused by human activity. We then use a parameter for the amount of erosion per thousand footsteps to approximate an average for the total number of steps that a stair has experienced. Though our model is unable to determine if the stairs were used primarily for going up or down, we can infer if multiple people used it simultaneously given the number of troughs in the erosion pattern and the total width of each step. Repairs or renovations may have occurred if given the amount of foot traffic, the amount of wear on the steps is too little. Assuming 1000 footsteps erode between 0.001 and 0.01 cm^3 of granite, we estimated that the steps in the Leaning Tower of Pisa endure an average of somewhere between 591,000 and 5.05 million steps per year. The Tower hosts approximately 5 million visitors every year [2], and assuming not all of them climb the stairs to the top but everyone who does goes both up and down, our estimate seems to be in the realm of plausibility.

Key Terms: quarter-ellipsoid, erosion, foot traffic, linear relation.

Contents

A Little Worse for Wear

January 27, 2025

1 Introduction

Stairs are one of the oldest architectural features in human history. As is the case for any structure, stairs face wear and degradation over time, both naturally and from human activity. As shown in **Figure 1**, we can observe this decay in the shape of a bow (see **Figure 1b**). Given a set of measurements from a set of stairs, our goal is to estimate the average amount of foot traffic the stairs have received over a certain number of years. To do this, we developed a mathematical model that incorporates the building material of the stair, the observed amount of stair lost, human foot traffic, and environmental degradation.

1.1 Terminology

Throughout this paper, we make several references to the different parts of a stair. We refer to the vertical distance between consecutive stairs as the *rise* of a stair and refer to the horizontal distance between stair risers as the *tread* of a stair (see **Figure 2**).

2 Assumptions

To demonstrate our model, we assume that the stairs in question are made out of granite, and the shape of the eroded area is one of a quarter-ellipsoid, as shown in **Figure 3**. Assuming the stairs are made out of granite, we know (roughly) that erosion happens at approximately one meter per million

(a) Staircase showing erosion from prolonged foot traffic and erosion.

(b) Close-up of bow erosion pattern on individual steps.

Figure 1: Bowing erosion pattern on a historical staircase [7].

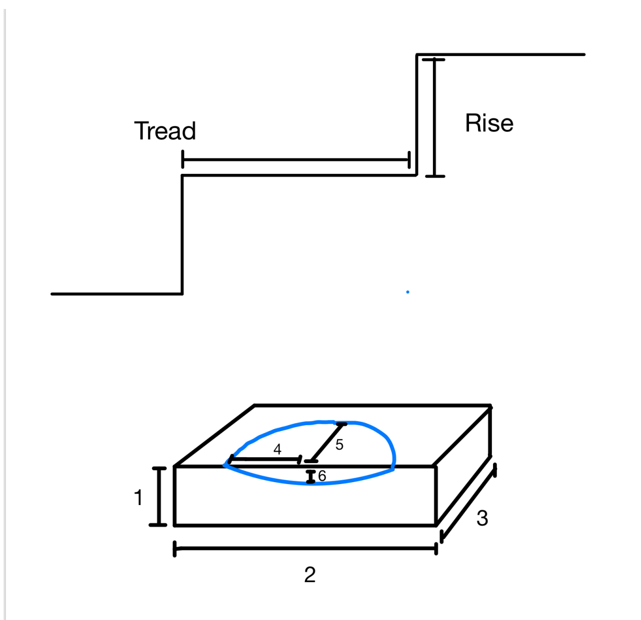

Figure 2: Stair anatomy and measurements taken. Note that the numbers do not represent lengths.

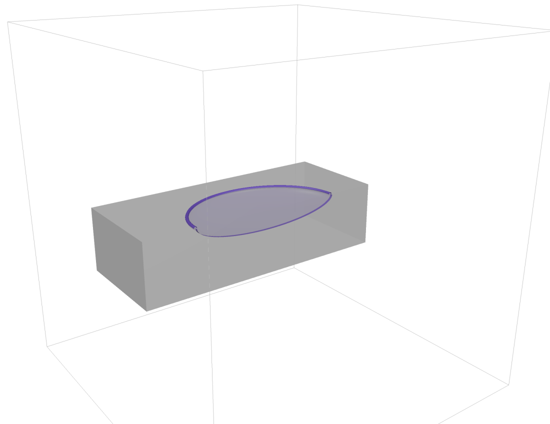

Figure 3: 3D model of the quarter-ellipsoid that approximates the volume of the eroded area.

years from only natural degradation due to the environment [1]. We also assume in this case that we have a rough estimate of the age of the staircase. We also assume that humans are likely to ”funnel” into places that have already been carved out because people naturally want to follow the path of least resistance. It is also assumed that the sides and back of each step are going to be the least eroded. Finally, we assume that every thousand steps on a stair step erodes between 0.001 cm^3 and 0.01 cm^3 of granite. These estimates come from us performing a similar calculation for a marble staircase as seen in section 5, and marble is approximately 3-4 levels lower on the Mohs hardness scale than granite [8][4].

3 Required Measurements

Our model requires several simple measurements that only need a measuring device, like a ruler or tape measure. First, the archaeologists need to calculate the volume of the entire step. To do so, they must measure the length, height, and depth of a stair. Height calculations should be done at one of the sides of the step, to get as close a measurement to the original height of the step as possible. These measurements will be referred to as H_o, W_o and D_o, respectively, or 1, 2, and 3 in **Figure 2**. Next, they need to measure the size of the eroded area. First, they should measure how deep the eroded area is at its lowest point. This can be done by measuring the distance from the bottom of the riser to the lowest point of the erosion and then subtracting from H_o. Next, measure the entire horizontal distance of the eroded area and divide it by two to find the distance from the center of the eroded area. Finally, measure the distance from the midpoint at the front of the tread to the edge of the eroded area perpendicular to W_o. These will be referred to as W_e, D_e and H_e, respectively, or 4, 5, and 6 in **Figure 2**.

3.1 List of Symbols

- H_o Total height of the stair (cm).
- W_o Total length of the stair (cm).
- D_o Total depth (tread) of the stair (cm).
- H_e Height from the center of the eroded area to the original rise of the stair (cm).
- W_e Width from the center of the eroded area to the edge (cm).
- D_e Depth from the center of the eroded area to the edge (cm).
- $V_{original}$ Original volume of the stair (cm^3).
- $V_{erosion}$ Volume of the eroded area of the stair (cm^3).
- V_{human} Erosion due to human activity (cm^3).
- $V_{natural}$ Erosion due to nature and the environment (cm^3).
- $E_{natural}$ Erosion due to nature (μm).
- H Human caused erosion per year ($\frac{\text{cm}^3}{\text{year}}$).
- α Age of the stair (years).
- β How much volume is eroded per 1000 steps (cm^3).
- T Total number of steps that a stair step has experienced.

4 Our Model

Now that we have the necessary measurements, we can calculate the original volume of a stair as

$$V_{stair} = H_o \cdot W_o \cdot D_o, \tag{1}$$

and the volume of the eroded area as

$$V_{eroded} = \frac{1}{4} \cdot \frac{4}{3}\pi \cdot W_e \cdot D_e \cdot H_e. \tag{2}$$

This formula comes from the formula for the volume of an ellipsoid, multiplied by $\frac{1}{4}$. Effectively, our measurements of the eroded area were simply the lengths of the semi-axes of the ellipsoid that model the erosion. Next, we know that granite naturally erodes at a rate of about 1 meter per million years [1], or 1 micrometer per year. Given that we know a rough estimate of the age of the stairs α, we can calculate how much erosion we expect due to natural causes with

$$V_{natural} = 1\mu m \cdot \alpha \tag{3}$$

To determine how much of the erosion was due to human activity, we can subtract the amount of natural erosion from the total to get

$$V_{human} = V_{eroded} - V_{natural} \tag{4}$$

Now we can divide by the age of the stairs α to get the amount of human-caused erosion per year,

$$H = \frac{V_{human}}{\alpha} \tag{5}$$

From our previous assumption that every thousand steps on a stair step erodes between 0.001cm^3 and 0.01cm^3 of granite, we can calculate the number of steps that have been taken on the stairs in total. Suppose the amount of erosion per thousand steps is β. Then,

$$T = \frac{V_{human}}{\beta} \tag{6}$$

So T is the total number of steps, and its value varies as β shifts in its range.

Figure 4: Stairs in the Leaning Tower of Pisa [6].

5 Example and Validation

The stairs leading to the top of the Leaning Tower of Pisa in Pisa, Italy are a prime example of long-term staircase erosion (see **Figure 4**). Note the consistent eroded area on the left side of each step. Construction of the tower began in 1173 and was completed with the addition of the bell chamber in 1372, although the topmost floor was completed in 1319 [2]. We can reasonably assume then that the stairs closer to ground level have been used more than the ones at the top, given the amount of time it took to complete the tower. Thus, a rough estimate for the age of the marble stairs is anywhere from 653 to 852 years old, with different sections of the staircase falling in different parts of this age range. For simplicity's sake, we use the extrema of this range to provide a lower and upper bound on the stairs' usage. We start by estimating the dimensions of each step of the staircase. The two main spiral staircases (one of which is pictured in **Figure 4**) that run the height of the tower are narrow and only one person can travel on them at a time, single-file [2]. In modern-day, building codes typically mandate a minimum of 36 inches of stair width per person traveling on the staircase, or 0.91 meters, though most single-file staircases are 44 inches wide, or 1.12 meters [3]. Since spiral stairs are usually tighter and modern building regulations didn't apply in the 12th century, we will assume these stairs are 1 meter wide. We will also assume typical riser height and tread depth here at 7 inches and 12 inches, or 0.1778 m and 0.3048 m respectively. Using these values and **Figure 4**, we can estimate the dimensions of the eroded area as follows:

$$H_e = 4 \text{ cm}$$

$$W_e = 37.5 \text{ cm}$$

$$D_e = 25 \text{ cm}$$

Plugging this into Equation (2), we get

$$\begin{aligned} V_{eroded} &= \frac{1}{4} \cdot \frac{4}{3}\pi \cdot H_e \cdot W_e \cdot D_e \\ &= \frac{1}{4} \cdot \frac{4}{3}\pi \cdot 4 \cdot 37.5 \cdot 25 \\ &= 3926.9905 \text{ cm}^3. \end{aligned}$$

Now, we need to calculate how much of this erosion was due to human activity and not natural degradation. We know that granite erodes at a rate of about 1 meter per million years [1], or 1 micrometer per year, but it is much harder than marble. Given that the marble stairs inside the tower are somewhat protected from the elements, we estimate that the marble naturally erodes at about 0.75 micrometers per year. We can now get a range for how much the marble eroded naturally with

$$E_{natural} = 0.75\mu\text{m} \cdot 653 = 489.75\mu\text{m}$$

$$E_{natural} = 0.75\mu\text{m} \cdot 852 = 639\mu\text{m}$$

or 0.0489 cm and 0.0639 cm per year. By using Equation (2) and replacing H_e with these values, we can get an estimate for how much volume was eroded naturally as follows

$$\begin{aligned} V_{natural} &= \frac{1}{4} \cdot \frac{4}{3}\pi \cdot 0.0489 \cdot 37.5 \cdot 25 \\ &= 48.00745 \text{ cm}^3. \end{aligned}$$

or

$$\begin{aligned} V_{natural} &= \frac{1}{4} \cdot \frac{4}{3}\pi \cdot 0.0639 \cdot 37.5 \cdot 25 \\ &= 62.7335 \text{ cm}^3. \end{aligned}$$

By using Equation (4), we determine a range for how much erosion was due to human activity,

$$\begin{aligned} V_{human} &= V_{eroded} - V_{natural} \\ &= 3926.9905 - 48.00745 \\ &= 3878.983 \text{ cm}^3, \end{aligned}$$

and

$$\begin{aligned} V_{human} &= V_{eroded} - V_{natural} \\ &= 3926.9905 - 62.7335 \\ &= 3864.257 \text{ cm}^3. \end{aligned}$$

Thus, we claim that given the age range of 653 to 852 years for the marble staircase in the Leaning Tower of Pisa, each stair step has between 3864.257 and 3878.983 cm^3 of marble eroded, out of a total

$$V_{original} = 1000 \cdot 17.78 \cdot 30$$
$$= 533,400 \text{ cm}^3$$

original marble volume per step, meaning approximately 0.724% to 0.727% of the volume of the stair step eroded. Finally, to get an estimate for the total number of steps that the stairs have endured, we can use our approximations for the amount of material that is eroded every 1000 steps. Because we are now iterating over a range for two variables, we turn to Python and SageMath with the following:

$$T = \frac{1000 \text{ steps}}{\beta} \cdot V_{human} \tag{7}$$

where T is the total number of steps over the lifetime of the stairs (see Figure 7 for code snippet). We iterate over β and V_{human} as described before, and plot the resulting surface in $\mathbb{R}^3$ (see Figure 5). This gives us an estimate of somewhere between 391 million and 3.8 billion total steps over the lifetime of the stairs. With an average age of 752.5 years, we arrive at 519,601 to 5.05 million steps per year.

6 Model Analysis

Our model allows us to make some deductions about certain characteristics that archaeologists would be interested in. We already can calculate total foot traffic and approximate annual foot traffic. It would also be possible to estimate the age of the stairs if we had an estimate for the total number of steps endured. However, it is not possible to determine whether a given staircase was favored for going only in a certain direction using our model. We were not able to find any qualities of stair steps that indicate a particular staircase was favored for one direction. Though our model assumes a symmetrical eroded area with one trough, we can extend our model and presume people walked side-by-side if there are two visible troughs (or more) on a wide enough set of stairs.

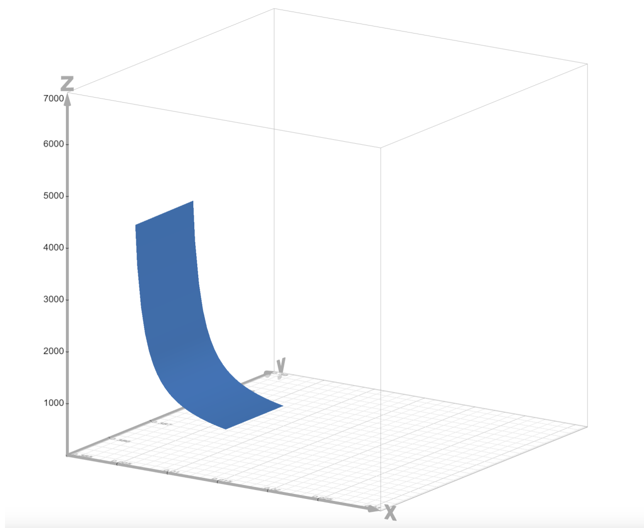

Figure 5: Graph of total steps as a function of human volume and eroded volume/1000 steps. The z-axis has been scaled due to limitations in Desmos.

With our model, it is possible to predict the approximate age of stairs given how much foot traffic there is per year, and predict how much foot traffic there is based on the age and amount of wear the stairs have undergone. Either a foot traffic estimate or time estimate must be given for the model to work, in addition to the current measurements.

Using the model, it would only be possible to predict if a stone or wood came from a certain quarry or tree if the material's hardness factor matched what we used in the model given the amount of time and people that had used the stairs. Therefore, our model would struggle to identify the source of the specific material unless more resources could be dedicated to physically analyzing the stairs.

To determine if a particular set of stairs has been repaired or renovated, we would have to use our model given age and average foot traffic estimates. This would allow us to work backward to determine the total amount of erosion we would expect on the stair. If the erosion on the staircases differs significantly from the estimate (i.e. there is not as much erosion as predicted), we can conclude that at some point the stair was repaired, renovated, or possibly even replaced.

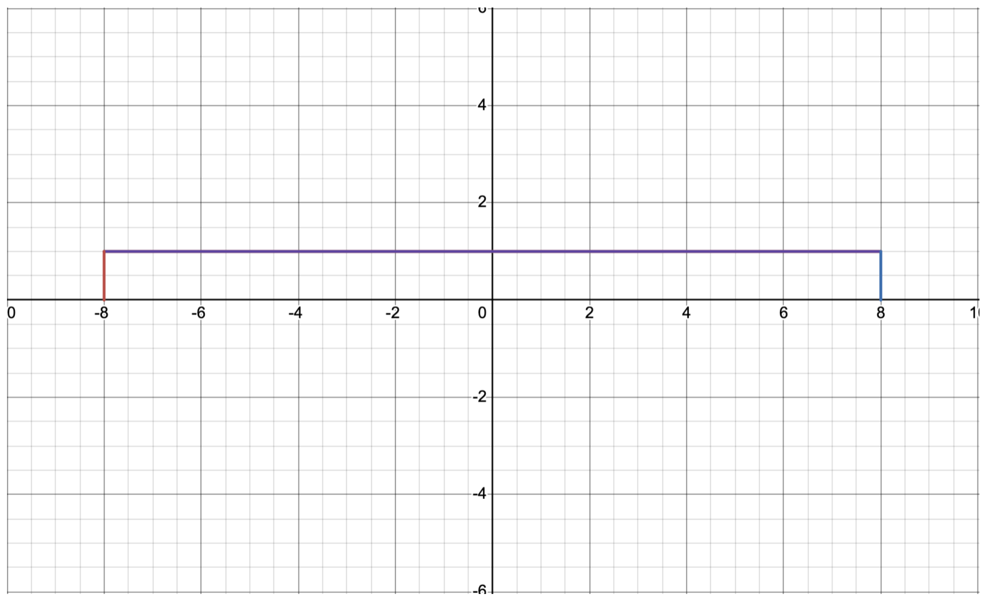

Figure 6a: Graph of Equation (8) when $b = e$.

7 Possible Improvements and Future Considerations

There are several improvements we could make to our model to extract more accurate information. First, the shape of the eroded areas, when looking at the face of the riser, appears more like inverted bell curves as opposed to a cross-section of an ellipsoid. We can model the shape of the face of the erosion with a modified equation representing a bell curve,

$$f(x) = -0.5\ln(\ln(b))e^{(-\frac{x^2}{b})} + 1 \tag{8}$$

where b is the standard deviation of the distribution represented by the curve. We multiply by -1 to invert the curve, multiply by a factor of $0.5\ln(\ln(b))$ to slow down the rate of the trough's descent as b increases, and add 1 for readability on a graph (see Figures 6a, 6b, and 6c). Note that due to the double natural log, b must be at least e to properly represent the erosion of the stair step over time.

We could also try a new approach to determining the volume of the eroded area by using water (or any fluid), a flat plastic board, and a thin plastic wrap. By placing a flat board on the face of the riser, we could turn the eroded area into a cavity, cover the bottom with plastic wrap to prevent it from getting wet and measure exactly the volume of water/fluid it takes to fill the cavity, giving us a precise

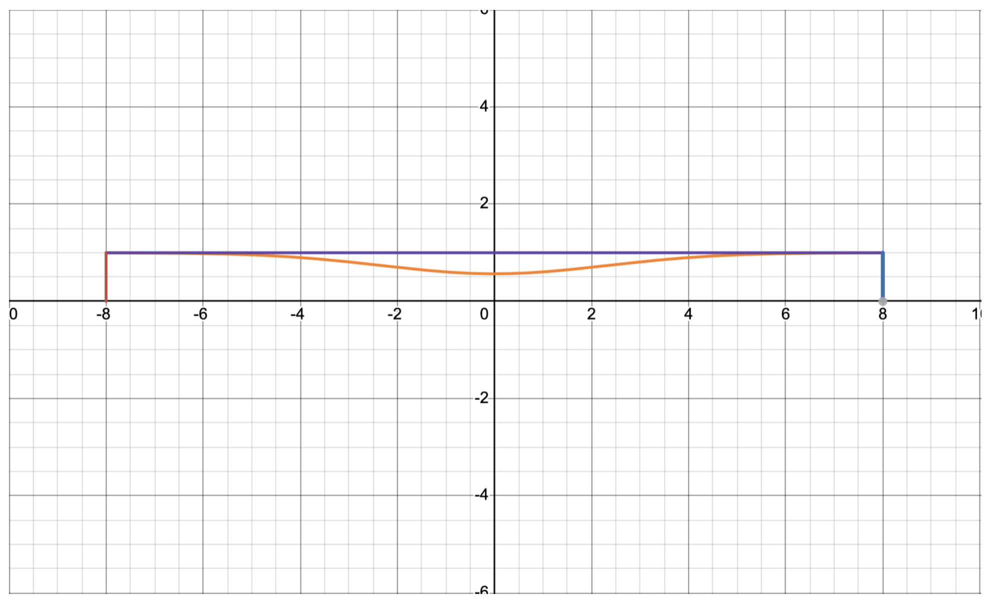

Figure 6b: Graph of Equation (8) when $b = 8 + e$.

Figure 6c: Graph of Equation (8) when $b = 33 + e$.

volume measurement. This method would certainly take more time, effort, and manpower, and special care would need to be taken to ensure the measurement is non-destructive.

Another potential consideration would be accounting for varying use of the stairs over time, particularly if they are from the ruins of an ancient civilization, or have become a modern-day tourist attraction that visitors can go on (or both). Certainly the usage of the stairs would not be consistent over hundreds or even thousands of years, affecting how our measurement of their erosion unveils how many people used them over history. This would require more information about the stairs themselves, including location, what areas the stairs connected, and the kind of societies and cultures that lived there, which would increase the cost of measurements and physical analysis of the stairs.

8 Conclusion

Overall our model predicts that for a set of stairs in the Leaning Tower of Pisa between 653 and 852 years old, each stair step that is 100 cm wide, with 17.78 cm of riser height and 30.48 cm of tread depth will have between 3864 and 3879 cubic centimeters of marble erosion due to human footsteps, which results in approximately 591,000 to 5.05 million steps per year over the life of the stairs. This linear model assumes a linear erosion process for both natural and human-induced erosion.

Our model can be used by archaeologists who may have an estimate of the age of a set of stairs and have taken appropriate measurements to compute how much the stairs were used (on average) over their lifetime. They can also deduce other characteristics such as how many people used the stairs simultaneously, by observing the number of troughs in the erosion pattern. Our model cannot be used to predict if people go primarily up or down.

8.1 Guidance to Archaeologists

Our recommendation to archaeologists studying old stairs would be to find the volume of the eroded area, estimate the age and amount eroded per 1000 steps, and calculate an approximation for the total number of steps endured. Our model gives a general picture of the usage of the stairs over time and can be made more precise (see section 7) with an increase in required resources.

9 References

References

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10 Appendix

```
import numpy as np

T(x, v) = (1000*v)/x                                # formula for total foot traffic

x_r = np.arange(0.001, 0.01, 0.0001, dtype=float)     # range for erosion per thousand steps
v_r = np.arange(3864.257, 3878.989, 0.1, dtype=float)   # range for amount of erosion

t = []

for x in x_r:
    for v in v_r:
        t.append((T(x,v), x, v))                    # compute over both ranges and append 3-tuple to list

print(t)
```

Figure 7: Code to iterate over parameter ranges for Equation 7